On a New n-norm and some Identical n-Norms On a Hilbert Space

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Abstract: In this paper, we discuss the concept of n-normed spaces and introduce a new n-norm. Further we show the equality of five formulae of n-norms on a Hilbert space and the equality of six formulae of n-norms on a separable Hilbert space. Also we prove the equality of two formulae of n-norms on the dual space of an n-normed space.

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1 Introduction

LET$\boldsymbol{X}$ be a real vector space with dim$\boldsymbol{X}$≥ n, where n is a positive integer. A real valued function $\|.,...,.\|: X^n \to \mathbf{R}$ is called an n-norm on X if the following conditions hold:

(1) $\|x_1,...,x_n\| = 0$iff $x_{1,...,}x_n$ are linearly dependent

(2) $\|x_1,...,x_n\|$ is invariant under permutations of $x_1,...,x_n$

(3) $\|\alpha x_1, x_2,...,x_n\| = |\alpha|\|x_1,...,x_n\|$ for any $\alpha \in \mathbf{R}$

(4) $\|x_0 + x_1, x_2,...,x_n\| \le \|x_0,...,x_n\| + \|x_1,...,x_n\|$ for all $x_0, x_1,...,x_n \in X$.

The pair $(X, \|.,...,.\|)$ is called an n-normed space. An n-norm is always non-negative. The combination of conditions (3) and (4) above gives the non-negativity of an n-norm. If X is an n-normed space with dual X'.the following formula (as formulated by Gähler [2])

$$\|x_1,...,x_n\|^G = \sup_{f_j \in X', \|f_j\| \le 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on X.

If X is equipped with an inner product $\langle .,. \rangle$, we can define the standard n-norm on X by $\|x_1,...,x_n\|^S = \sqrt{\det[\langle x_i, x_j \rangle]}$.

Note that the value of $\|x_1,...,x_n\|^S$ represents the volume of n-dimensional parallelepiped spanned by $x_1,...,x_n$.Let X be a Hilbert space with dual $\boldsymbol{X}'$.ThenGähler's formula on X becomes

$$\|x_1,...,x_n\|^G = \min_{y_j \in X, \|y_j\| \le 1} det[\langle x_i, y_j \rangle]$$

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Also the function

$$\|x_1,...,x_n\|^D = \sup_{y_j \in X, \|y_1,...,y_n\|^S \le 1} \begin{vmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix}$$

defines an n-norm on a Hilbert space $\boldsymbol{X}$.Then $\|.,...,.\|^G$ and $\|.,...,.\|^D$ are identical on a Hilbert space X[**6**].

If $\boldsymbol{X}$ is a separable Hilbert space and $\{e_1, e_2, ...\}$ is a complete orthonormal set in $\boldsymbol{X}$, we can define an n-norm on $\boldsymbol{X}$by

$$\|x_1,...,x_n\|_2 = \left[\frac{1}{n!}\sum_{j_1} \cdots \sum_{j_n} \left|det\left[\alpha_{ij_k}\right]\right|^2\right]^{\frac{1}{2}}$$

Where $\alpha_{ij} = \langle x_i, e_j \rangle$[5],[6].

Then $\|.,...,.\|^G$, $\|.,...,.\|^S$, $\|.,...,.\|_2$ and $\|.,...,.\|^D$ are identical on a separable Hilbert space [6].

The theory of 2-normed spaces and n-normed spaces were initially developed by Gähler[1],[2],[3],[4]in the 1960's.Recent works and related works can be found in [5],[6],[8],[11].Our interest here is to study alternative formulae of n-norms especially in a Hilbert space. The alternative formulae are identical with the four forms of n-norms mentioned above. In the last part we study the equality of two n-norms defined on the dual space of an n-normed space.

2 A New n-Norm

Proposition 2 .1: Let$\boldsymbol{X}$ be a real vector space with dim $\boldsymbol{X} \ge n$ equipped with an inner product $\langle .,. \rangle$. Then the function $\|x_1,...,x_n\|^R =$

$$Abs\left(\begin{vmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix}\right)$$

$$= \left|det[\langle x_i, y_j \rangle]\right|$$

defines an n-norm on $\boldsymbol{X}$ for fixed linearly independent n elements $y_1, y_2, ..., y_n \in X$.

Proof: (i) $x_1, x_2, ..., x_n$ are linearly dependent

⇔the rows of the matrix $[\langle x_i, y_j \rangle]$ are linearly dependent for fixed linearly independent elements $y_1, y_2, ..., y_n \in X$

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$$\Leftrightarrow det[\langle x_i, y_j\rangle] = 0$$
$$\Leftrightarrow |det[\langle x_i, y_j\rangle]| = 0$$
$$\Leftrightarrow \|x_1, \dots, x_n\|^R = 0$$

(ii) The value of $|det[\langle x_i, y_j\rangle]|$ remains unaltered under the interchange of rows (or columns)

$\Rightarrow \|x_1, \dots, x_n\|^R$remains invariant under permutations of $x_1, x_2, \dots, x_n$.

(iii) For any $\alpha \in \mathbb{R}$,

$$\|\alpha x_1, \dots, x_n\|^R = Abs\left(\begin{vmatrix}\langle \alpha x_1, y_1\rangle & \cdots & \langle \alpha x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}\right)$$

$$= Abs\left(\begin{vmatrix}\alpha\langle x_1, y_1\rangle & \cdots & \alpha\langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}\right)$$

$$= Abs\left(\alpha\begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}\right)$$

$$= |\alpha| Abs\left(\begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}\right)$$

$$= |\alpha| \|x_1, \dots, x_n\|^R$$

(iv) For arbitrary elements $x_1', x_1, x_2, \dots, x_n \in X$,

$$\begin{vmatrix}\langle x_1' + x_1, y_1\rangle & \cdots & \langle x_1' + x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$
$$= \begin{vmatrix}\langle x_1', y_1\rangle + \langle x_1, y_1\rangle & \cdots & \langle x_1', y_n\rangle + \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$
$$= \begin{vmatrix}\langle x_1', y_1\rangle & \cdots & \langle x_1', y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix} + \begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$

$$\Rightarrow Abs\left(\begin{vmatrix}\langle x_1'+x_1, y_1\rangle & \cdots & \langle x_1' + x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}\right)$$
$$\leq Abs\left(\begin{vmatrix}\langle x_1', y_1\rangle & \cdots & \langle x_1', y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}\right)$$
$$+ Abs\left(\begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \ddots & \langle x_n, y_n\rangle\end{vmatrix}\right)$$

$$\therefore \|x_1' + x_1, \dots, x_n\|^R \leq \|x_1', \dots, x_n\|^R + \|x_1, \dots, x_n\|^R$$

This completes the proof.

3 Various Alternative Formulae Of n-Norms And Their Equality:

Proposition 3.1: The function

$$\|x_1, \dots, x_n\|^E = \operatorname*{Sup}_{y_j \in X, \|y_1, \dots, y_2\|^S = 1}\begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$

defines an n-norm on a Hilbert space X.

Proof: (i) $x_, x_2, \dots x_n$ are linearly dependent

$\Leftrightarrow$the rows of the matrix $[\langle x_l, y_j\rangle]$ are linearly dependent for all $y_1, y_2, \dots, y_n \in X$ with $\|y_1, y_2, \dots, y_n\|^S = 1$

$\Leftrightarrow det[\langle x_i, y_j\rangle] = 0 \Leftrightarrow \|x_1, \dots, x_n\|^E = 0.$

(ii) By the properties of determinant, $\|x_1, \dots, x_n\|^E$ is invariant under permutations of $x_1, x_2, \dots, x_n$.

(iii) For any$\alpha \in \mathbb{R}$,

$$\|\alpha x_1, \dots, x_n\|^E = \sup_{y_j \in X, \|y_1, \dots, y_n\|^S = 1}\begin{vmatrix}\langle \alpha x_1, y_1\rangle & \cdots & \langle \alpha x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$
$$= |\alpha| \sup_{y_j \in X, \|y_1, \dots, y_n\|^S = 1}\begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$
$$= |\alpha| \|x_1, \dots, x_n\|^E$$

(iv) For arbitrary elements $x_1', x_1, x_2, \dots, x_n \in X$,

$$\begin{vmatrix}\langle x_1' + x_1, y_1\rangle & \cdots & \langle x_1' + x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$
$$= \begin{vmatrix}\langle x_1', y_1\rangle + \langle x_1, y_1\rangle & \cdots & \langle x_1', y_n\rangle + \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$
$$= \begin{vmatrix}\langle x_1', y_1\rangle & \cdots & \langle x_1', y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix} + \begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$

Takingsupremums on both sides, we have

$\|x_1' + x_1, \dots, x_n\|^E \leq \|x_1', \dots, x_n\|^E + \|x_1, \dots, x_n\|^E.$

This completes the proof.

Proposition 3.2: $\|., \dots, .\|^D$ and $\|., \dots, .\|^E$ are identical on a Hilbert space X.

Proof: Obviously,

$$\|x_1, \dots, x_n\|^E = \sup_{y_j \in X, \|y_1, \dots, y_n\|^S = 1}\begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$
$$\leq \sup_{y_j \in X, \|y_1, \dots, y_n\|^S \leq 1}\begin{vmatrix}\langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle\end{vmatrix}$$
$$= \|x_1, \dots, x_n\|^D$$

Conversely, define

$z_j = \frac{y_j}{\sqrt[n]{\|y_1, \dots, y_n\|^S}} = \frac{y_j}{a}forj = 1, 2, \dots, n$where $a = \sqrt[n]{\|y_1, , \dots, y_n\|^S}$.

Then,

$$\begin{vmatrix} \langle x_1, y_1\rangle & \cdots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \cdots & \langle x_n, y_n\rangle \end{vmatrix} = \begin{vmatrix} \langle x_1, az_1\rangle & \cdots & \langle x_1, az_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, az_1\rangle & \cdots & \langle x_n, az_n\rangle \end{vmatrix}$$

$$= a^n \begin{vmatrix} \langle x_1, z_1\rangle & \cdots & \langle x_1, z_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1\rangle & \cdots & \langle x_n, z_n\rangle \end{vmatrix}$$

$$\leq \operatorname*{Sup}_{z_j \in X, \|z_1,\dots,z_n\|^S = 1} a^n \begin{vmatrix} \langle x_1, z_1\rangle & \cdots & \langle x_1, z_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1\rangle & \cdots & \langle x_n, z_n\rangle \end{vmatrix}$$

$$= a^n \sup_{z_j \in X, \|z_1,\dots,z_n\|^S = 1} \begin{vmatrix} \langle x_1, z_1\rangle & \cdots & \langle x_1, z_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1\rangle & \cdots & \langle x_n, z_n\rangle \end{vmatrix}$$

$$\leq \sup_{z_j \in X, \|z_1,\dots,z_n\|^S = 1} \begin{vmatrix} \langle x_1, z_1\rangle & \cdots & \langle x_1, z_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1\rangle & \cdots & \langle x_n, z_n\rangle \end{vmatrix}$$

$= \|x_1, \dots, x_n\|^E$ for all $y_j \in X$ with $\|y_1, \dots, y_n\|^S \leq 1$.

$\therefore \|.,\dots,.\|^D$ and $\|.,\dots,.\|^E$ are identical.

Proposition 3.3: $\|.,\dots,.\|^G$ and $\|.,\dots,.\|^E$ are identical on a Hilbert space X.

Proof: $\|.,\dots,.\|^G$ and $\|.,\dots,.\|^D$ are identical on X[**6**]. But $\|.,\dots,.\|^D$ and $\|.,\dots,.\|^E$ are identical [above proposition]. This completes the proof.

Corollary 3.1: $\|.,\dots,.\|^G$, $\|.,\dots,.\|^S$, $\|.,\dots,.\|^D$ and $\|.,\dots,.\|^E$ are identical.

Proposition 3.4: On a separable Hilbert space X, $\|.,\dots,.\|^E$ and $\|.,\dots,.\|_2$ are identical.

Proof: Let $\{e_1, e_2, \dots\}$ be a complete orthonormal set in X. Then, $\|x_1, \dots, x_n\|_2$ may be derived directly from standard n-norm $\|x_1, \dots, x_n\|^S$[**6**] $\Longrightarrow \|.,\dots,.\|_2$ and $\|.,\dots,.\|^S$ are identical. But, $\|.,\dots,.\|^S$ and $\|.,\dots,.\|^E$ are identical. So, $\|.,\dots,.\|^E$ and $\|.,\dots,.\|_2$ are identical.

Corollary 3.2: On a separable Hilbert space X, $\|.,\dots,.\|^D$, $\|.,\dots,.\|^S$, $\|.,\dots,.\|_2$, $\|.,\dots,.\|^G$ and $\|.,\dots,.\|^E$ are identical.

Proposition 3.5: Let X be a normed space with dual X' . Then the function

$$\|x_1, \dots, x_n\|^r = \sup_{f_j \in X', \|f_j\| = 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

$$= \sup_{f_j \in X', \|f_j\| = 1} det[f_j(x_i)]$$

defines an n-norm on X.

Proof: (i) Obviously, $x_1, \dots, x_n$ are linearly dependent

- $\Leftrightarrow$ rows of the matrix $[f_j(x_i)]$ are linearly dependent.
- $\Leftrightarrow$ $\det[f_j(x_i)] = 0\ \forall f_j \in X' with\ \|f_j\| = 1$
- $\Leftrightarrow$ $\sup_{f_j \in X', \|f_j\| = 1} \det[f_j(x_i)] = 0$
- $\Leftrightarrow$ $\|x_1, \dots, x_n\|^r = 0\ \forall\ x_1, \dots, x_n \in X$.

(ii) By the properties of determinants, $\|x_1, \dots, x_n\|^r$ remains invariant under permutations of $x_1, \dots, x_n$.

(iii) $\forall\ \alpha \in \mathbb{R}$,

$$\|\alpha x_1, \dots, x_n\|^r = \operatorname*{Sup}_{f_j \in X', \|f_j\| = 1} \begin{vmatrix} f_1(\alpha x_1) & \dots & f_n(\alpha x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$$

$$= \operatorname*{Sup}_{f_j \in X', \|f_j\| = 1} \alpha \begin{vmatrix} f_1(x_1) & \dots & f_n((x_1)) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$$

$$= |\alpha| \operatorname*{Sup}_{f_j \in X', \|f_j\| = 1} \begin{vmatrix} f_1(x_1) & \dots & f_n((x_1)) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$$

$$= |\alpha| \|x_1, \dots, x_n\|^r$$

(iv) By the linearity of f_j's and the properties of determinants, $\begin{vmatrix} f_1(x_0 + x_1) & \dots & f_n((x_0 + x_1)) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$

$$= \begin{vmatrix} f_1(x_0) & \dots & f_n((x_0)) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix} + \begin{vmatrix} f_1(x_1) & \dots & f_n((x_1)) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$$

Taking supremums of both sides over $f_j \in X'$ with $\|f_j\| = 1$, we have

$$\|x_0 + x_1, \dots, x_n\|^r \leq \|x_0, \dots, x_n\|^r + \|x_1, \dots, x_n\|^r$$

This completes the proof.

Proposition 3.6: Let $\boldsymbol{X}$ be a normed space with dual X'. Then, $\|.,\dots,.\|^G$ and $\|.,\dots,.\|^r$ are identical on X.

Proof: Obviously, $\|x_1, \dots, x_n\|^r \leq \|x_1, \dots, x_n\|^G$

Conversely, define $g_j = \frac{f_j}{\|f_j\|}\ for\ j = 1,2, \dots, n$.

Clearly $g_j \in X'$.

Now, $\begin{vmatrix} f_1(x_1) & \dots & f_n((x_1)) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$

$$= \begin{vmatrix} \|f_1\| g_1(x_1) & \dots & \|f_n\| g_n((x_1)) \\ \vdots & \ddots & \vdots \\ \|f_1\| g_1(x_n) & \dots & \|f_n\| g_n(x_n) \end{vmatrix}$$

$$= \|f_1\| \|f_2\| \dots \|f_n\| \begin{vmatrix} g_1(x_1) & \dots & g_n((x_1)) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \dots & g_n(x_n) \end{vmatrix}$$

$$\leq \sup_{g_j \in X', \|g_j\| = 1} \begin{vmatrix} g_1(x_1) & \dots & g_n((x_1)) \\ \vdots & \ddots & \vdots \\ g_1(x_n) & \dots & g_n(x_n) \end{vmatrix} [\because \|f_j\| \leq 1 \forall\ j]$$

$= \|x_1, \dots, x_n\|^r\ \forall f_j \in X' with\ \|f_j\| \leq 1$.

$\Rightarrow \|x_1, \dots, x_n\|^G \leq \|x_1, \dots, x_n\|^r$.

$\|x_1, \dots, x_n\|^G$ and $\|x_1, \dots, x_n\|^r$ are identical. This completes the proof.

NOTE: if X is a Hilbert space with dual X', then $\|x_1, \dots, x_n\|^r$ becomes

$$\|x_1, \dots, x_n\|^r = \sup_{y_j \in X, \|y_j\| = 1} det[\langle x_i, y_j\rangle]$$

It follows from:

By **Riesz-representation theorem**, for each bounded linear functional $f_j \in X' \; \exists \; y_j \in X$ such that $f_j(x_i) = \langle x_i, y_j \rangle$ and $\|f_j\| = \|y_j\|$.

Then, $\|x_1, \dots, x_n\|^r = \sup_{y_j \in X, \|y_j\| = 1} \begin{vmatrix} \langle x_1, y_1\rangle & \dots & \langle x_1, y_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1\rangle & \dots & \langle x_n, y_n\rangle \end{vmatrix}$

Proposition 3.7 : On a Hilbert space X, $\|.,\dots,.\|^r$ and the standard n – norm $\|.,\dots,.\|^s$ are identical.

Proof : $\|x_1, \dots, x_n\|^r = \mathrm{Sup}_{y_j \in X, \|y_j\|=1} \, det[\langle x_i, y_j\rangle]$

By **generalized Cauchy – Schwarz inequality** [**10**], we have

$$\det[\langle x_i, y_j\rangle] \le \sqrt{det[\langle x_i, x_j\rangle]}\sqrt{det[\langle y_i, y_j\rangle]}$$

$$\Rightarrow \sup_{y_j \in X, \|y_j\| = 1} det[\langle x_i, y_j\rangle] \le \sup_{y_j \in X, \|y_j\| = 1} \sqrt{det[\langle x_i, x_j\rangle]}\sqrt{det[\langle y_i, y_j\rangle]}$$

$$\Longrightarrow \|x_1, \dots, x_n\|^r \le \sup_{y_j \in X, \|y_j\| = 1} \|x_1, \dots, x_n\|^S \|y_1, \dots, y_n\|^s$$

But $\|y_1, \dots, y_n\|^s = \sqrt{det[\langle y_i, y_j\rangle]} \le \sqrt{\langle y_1, y_1\rangle\langle y_2, y_2\rangle \dots \langle y_n, y_n\rangle}$

[By Hadamard determinant theorem] [9]

$$= \|y_1\|\|y_2\| \dots \|y_n\|$$

$$\Rightarrow \|y_1, \dots, y_n\|^s \le \|y_1\|\|y_2\| \dots \|y_n\|$$

$$\Rightarrow \sup_{y_j \in X, \|y_j\| = 1} \|y_1, \dots, y_n\|^s \le 1$$

$$\therefore \|x_1, \dots, x_n\|^r \le \|x_1, \dots, x_n\|^s$$

Conversely, we assume $x_1', x_2', \dots, x_n'$ are linearly independent vectors.Let $x_1, x_2, \dots, x_n$ be vectors obtained from $x_1', x_2', \dots, x_n'$ through **Gram-Schmidthorthogonalisation process**. Then,

$$\|x_1\|\|x_2\| \dots \|x_n\| = \sqrt{\|x_1\|^2\|x_2\|^2 \dots \|x_n\|^2}$$

$$= \sqrt{det[\langle x_i, x_j\rangle]}$$

[$\because x_1, x_2, \dots, x_n$ are orthogonal.]

$$= \|x_1, x_2, \dots x_n\|^s.$$

If $y_j = \frac{x_j}{\|x_j\|} \; for \; j = 1,2, \dots, n$ we have

$$\begin{vmatrix} \langle x_1, y_1\rangle \cdots \langle x_1, y_n\rangle \\ \vdots \quad \ddots \quad \vdots \\ \langle x_n, y_1\rangle \cdots \langle x_n, y_n\rangle \end{vmatrix} = \begin{vmatrix} \langle x_1, \frac{x_1}{\|x_1\|}\rangle & \cdots & \langle x_1, \frac{x_n}{\|x_n\|}\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, \frac{x_1}{\|x_1\|}\rangle & \cdots & \langle x_n, \frac{x_n}{\|x_n\|}\rangle \end{vmatrix}$$

$$= \frac{1}{\|x_1\| \dots \|x_n\|} \|x_1\|^2 \dots \|x_n\|^2$$

$$= \|x_1\| \dots \|x_n\|$$

$$\Rightarrow det[\langle x_i, y_j\rangle] = \|x_1\| \dots \|x_n\| = \|x_1, \dots, x_n\|^s$$

$$\Rightarrow \sup_{y_j \in X, \|y_j\| = 1} det[\langle x_i, y_j\rangle] \ge \|x_1, \dots, x_n\|^s$$

$$\therefore \|x_1, \dots, x_n\|^r \ge \|x_1, \dots, x_n\|^s$$

Hence, $\|.,\dots,.\|^r$ and $\|.,\dots,.\|^s$ are identical.

This completes the proof.

Corollary 3.3 : $\|.,\dots,.\|^G$, $\|.,\dots,.\|^r$, $\|.,\dots,.\|^D$ & $\|.,\dots,.\|^E$ and $\|.,\dots,.\|^S$ are identical on a Hilbert space X.

Proof: $\|.,\dots,.\|^G$, $\|.,\dots,.\|^S$, $\|.,\dots,.\|^D$ & $\|.,\dots,.\|^E$ are identical **[by cor. (3.1)]**. But $\|.,\dots,.\|^r$ & $\|.,\dots,.\|^S$ are identical **[by proposition (3.7)].** This completes the proof.

Corollary 3.4: On a separable Hilbert space X, the six formulae of n-norms viz. $\|.,\dots,.\|^G$, $\|.,\dots,.\|^S$, $\|.,\dots,.\|^r$, $\|.,\dots,.\|_2$, $\|.,\dots,.\|^D$ & $\|.,\dots,.\|^E$ are identical.

Proof: The proof follows from:

$\|.,\dots,.\|_2$ & $\|.,\dots,.\|^S$ are identical on a separable Hilbert space and **[cor.3.3].**

4 N-Norms On Dual Space And Their Equality:

Let $(X\|.,\dots,.\|)$ be an n-normed space. The function

$\|.,\dots,.\|' : (x')^n \to \mathbb{R}$ given by

$$\|f_1, \dots, f_n\|' = \sup_{x_i \in X, \|x_1, \dots, x_n\| \le 1} \begin{vmatrix} f_1(x_1) \cdots f_n(x_1) \\ \vdots \quad \ddots \quad \vdots \\ f_1(x_n) \cdots f_n(x_n) \end{vmatrix}$$

defines an n-norm on $\boldsymbol{X'}$ **[6] .**

Proposition 4.1: Let $(X, \|.,\dots,\|)$ be an n-normed space. The function $\|.,\dots,.\|'_1 : (X')^n \to \mathbb{R}$ given by

$$\| f_1, \dots, f_n \|'_1 = \sup_{x_i \in X, \|x_1, \dots, x_n\| = 1} \begin{vmatrix} f_1(x_1) \cdots f_n(x_1) \\ \vdots \quad \ddots \quad \vdots \\ f_1(x_n) \cdots f_n(x_n) \end{vmatrix}$$

defines an n-norm on $\boldsymbol{X'}$.

Proof:(i) It is easy to show that $f_1, \dots, f_n$ are linearly dependent iff $\| f_1, \dots, f_n \|'_1 = 0$.

(ii) By the properties of determinant and definition of supremum, $\| f_1, \dots, f_n \|'_1$ is invariant under permutations of $f_1, \dots, f_n$.

(iii) Furthermore, for any $\alpha \in \mathbb{R}$,

$$\sup_{x_i \in X, \|x_1, \dots, x_n\| = 1} \begin{vmatrix} \alpha f_1(x_1) \cdots \alpha f_n(x_1) \\ \vdots \quad \ddots \quad \vdots \\ f_1(x_n) \cdots f_n(x_n) \end{vmatrix}$$

$$= |\alpha| \sup_{x_i \in X, \|x_1, \dots, x_n\| = 1} \begin{vmatrix} f_1(x_1) \cdots f_n(x_1) \\ \vdots \quad \ddots \quad \vdots \\ f_1(x_n) \cdots f_n(x_n) \end{vmatrix}$$

$\therefore \| \alpha f_1, \dots, f_n \|'_1 = |\alpha| \, \| f_1, \dots, f_n \|'_1 \; for\ any\ \alpha \in \mathbb{R}$

(iv) Also,

$$\begin{vmatrix} (f_0+f_1)(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ (f_0+f_1)(x_n) & \cdots & f_n(x_n) \end{vmatrix} = \begin{vmatrix} f_0(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ (f_0(x_n) & \cdots & f_n(x_n) \end{vmatrix} + \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

Taking supremums of both sides over x_i's $\in X$ with $\|x_1, \dots, x_n\| = 1$, we have

$$\| f_0 + f_1, \dots, f_n \|'_1 \leq \| f_0, \dots, f_n \|'_1 + \| f_1, \dots, f_n \|'_1.$$

This completes the proof.

Proposition 4.2: Let $(X, \|., \dots, .\|)$ be an n-normed space. Then $\|., \dots, .\|'$ and $\|., \dots, .\|'_1$ are identical on X' where X' is the dual X.

Proof: Obviously,

$$\| f_1, \dots, f_n \|'_1 = \sup_{x_i \in X, \|x_1, \dots, x_n\| = 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

$$\leq \sup_{x_i \in X, \|x_1, \dots, x_n\| \leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

$$= \|f_1, \dots, f_n\|' \Rightarrow \| f_1, \dots, f_n \|'_1 \leq \|f_1, \dots, f_n\|'$$

Conversely, define $y_i = \frac{x_i}{a}$,where $a = \sqrt[n]{\|x_1, \dots, x_n\|}$ and $0 < a \leq 1$.

Then

$$\begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} = \begin{vmatrix} f_1(ay_1) & \cdots & f_n(ay_1) \\ \vdots & \ddots & \vdots \\ f_1(ay_n) & \cdots & f_n(ay_n) \end{vmatrix}$$

$$\leq \sup_{y_i \in X, \|y_1, \dots, y_n\| = 1} a^n \begin{vmatrix} f_1(y_1) & \cdots & f_n(y_1) \\ \vdots & \ddots & \vdots \\ f_1(y_n) & \cdots & f_n(y_n) \end{vmatrix}$$

$$\leq \sup_{y_i \in X, \|y_1, \dots, y_n\| = 1} \begin{vmatrix} f_1(y_1) & \cdots & f_n(y_1) \\ \vdots & \ddots & \vdots \\ f_1(y_n) & \cdots & f_n(y_n) \end{vmatrix}$$

$$= \|f_1, \dots, f_n\|'_1 \ \forall x_i \in X \text{ with } \|x_1, \dots, x_n\| \leq 1$$

$$\Rightarrow \|f_1, \dots, f_n\|' \leq \|f_1, \dots, f_n\|'_1$$

$\therefore$ $\|., \dots, .\|'_1$ and $\|., \dots, .\|'$ are identical on X'. This completes the proof.

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